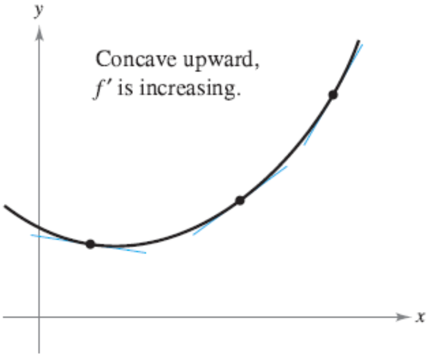
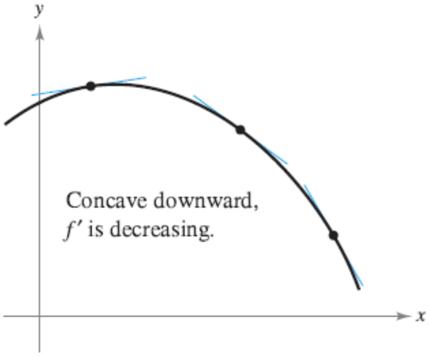
AP Calculus BC Sec 3.4 Concavity and the Second Derivative Test

DEFINITION OF CONCAVITY

Let f be differentiable on an open interval I. The graph of f is **concave upward** on I if f' is increasing on the interval and **concave downward** on I if f' is decreasing on the interval.





So besides finding a whole bunch of slopes to a curve to determine + or -, there has to be a better way...

THEOREM 3.7 TEST FOR CONCAVITY

Let f be a function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

ex. Determine the open intervals on which the graph is concave up or down. To do this - take $f''(x)$, set $= 0$, solve for x. Test points on either side of x into 2nd der. If $-$, concave down. If $+$, concave up.

$f(x) = x^3 + 3x^2 - 8$

$f'(x) = 3x^2 + 6x$

$f''(x) = 6x + 6 = 0$

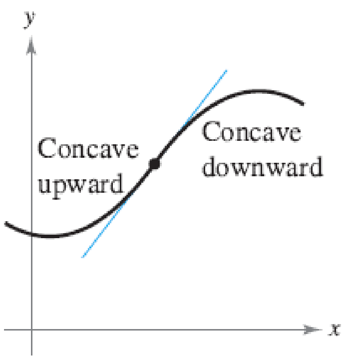
$x = -1$

$f''(x)$ $-$ | $+$

$x = -2 \quad -1 \quad 0$

concave down at $(-\infty, -1)$

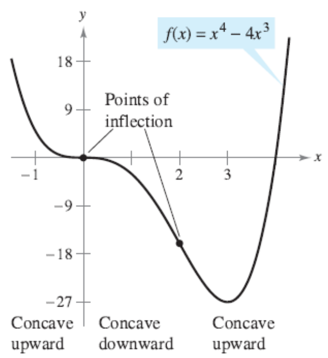
concave up $(-1, \infty)$



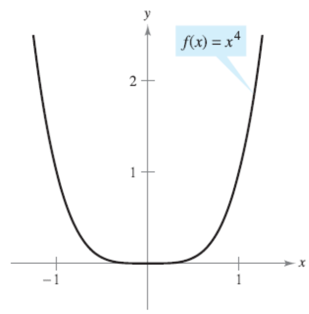
There is an instance where the graph is neither concave up or down... it is where the graph changes from concave up to down or from down to up. This is called a **point of inflection**.

THEOREM 3.8 POINTS OF INFLECTION

If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or f'' does not exist at $x = c$.



Points of inflection can occur where $f''(x) = 0$ or f'' does not exist.



$f''(x) = 0$, but $(0, 0)$ is not a point of inflection.

Now you try:

Find the points of inflection and discuss the concavity of the graph of the function.

$f(x) = 2x^4 - 8x + 3$

$f'(x) = 8x^3 - 8$

$f''(x) = 24x^2 = 0$

$x^2 = 0$

$x = 0$

$f''(x)$: $+$ | $+$

$x =$ -1, 0, 1

Since both are +, graph is concave up: $(-\infty, \infty)$

The points of inflection leads to the Second Derivative Test

THEOREM 3.9 SECOND DERIVATIVE TEST

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

1. If $f''(c) > 0$, then f has a relative minimum at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a relative maximum at $(c, f(c))$.

If $f''(c) = 0$, the test fails. That is, f may have a relative maximum, a relative minimum, or neither. In such cases, you can use the First Derivative Test.

Note: Taking CP from 1st der and putting in 2nd der.
If answer is >0, then min
If answer is <0, then max

Lets pull it all together with an example:

Find the relative extrema for: $f(x) = -3x^5 + 5x^3$ and sketch the graphs of all 3 functions.

To find graph of $f(x)$

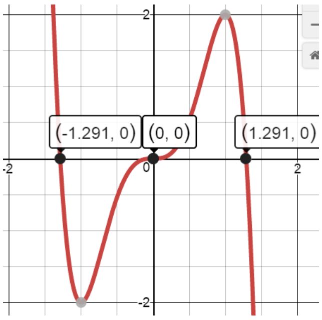
x-int: $-3x^5 + 5x^3 = 0$

$x^3(-3x^2 + 5) = 0$

$x = 0 \quad x^2 = \frac{5}{3}$

$x = \pm\sqrt{\frac{5}{3}} \approx \pm 1.3$ x-int

y-int: $y = 0$



$f'(x) = -15x^4 + 15x^2 = 0$

$-15x^2(x^2 - 1) = 0$

$x = 0, 1, -1$ CP where f(x) has a possible max/min

also x-int for f'(x)

x= -2, -.5, .5, 2

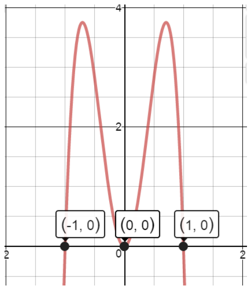
f'(x)= − −1 + 0 + 1 −

min max

so at (−1, −2) min

(1, 2) max

put back into f(x) to get actual point



$f''(x) = -60x^3 + 30x = 0$

$-30x(2x^2 - 1) = 0$

$x = 0, -\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}$ P of I for f(x)

but CP for f'(x), x-int for f''(x)

x= -1, -.5, .5, 1

f''(x) + $-\sqrt{\frac{1}{2}}$ − 0 + $\sqrt{\frac{1}{2}}$ −

f'(x) this means that $\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}$ is max

0 is min

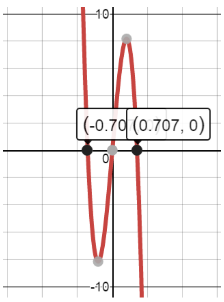
2nd der test: Take CP and put in f''(x)

$f''(-1) > 0$ min

$f''(1) < 0$ max

$f''(0) = 0$ test fails

neither max/min



Recap:

1st der: gave CP: −1, 0, 1

with chart we found

that x = −1 min

x = 1 max

x = 0 —

2nd der: P of I: $\pm\sqrt{\frac{1}{2}}, 0$

with chart we found

concave up: $(-\infty, -\sqrt{\frac{1}{2}}) \cup (0, \sqrt{\frac{1}{2}})$

concave down $(-\sqrt{\frac{1}{2}}, 0) \cup (\sqrt{\frac{1}{2}}, \infty)$