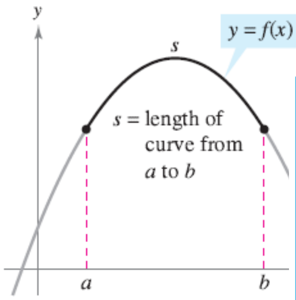
AP Calculus BC: Arc Length and Surface Area



DEFINITION OF ARC LENGTH

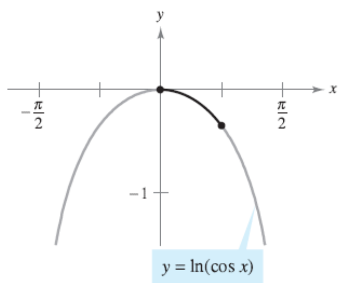
Let the function given by $y = f(x)$ represent a smooth curve on the interval $[a, b]$. The **arc length** of f between a and b is

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Similarly, for a smooth curve given by $x = g(y)$, the **arc length** of g between c and d is

$$s = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy.$$

Find the arc length of the graph of $\boxed{y = \ln(\cos x)}$ from $x = 0$ to $x = \pi/4$.



$$y = \ln(\cos x)$$

$$\frac{dy}{dx} = \frac{1}{\cos x}(-\sin x) = \frac{-\sin x}{\cos x}$$

$$= -\tan x$$

$$S = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$S = \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx$$

$$S = \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx$$

$$S = \int_0^{\pi/4} \sec x\, dx$$

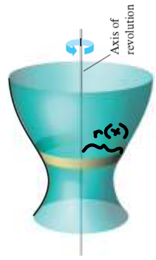
$$S = \Big[\ln|\sec x + \tan x|\Big]_0^{\pi/4}$$

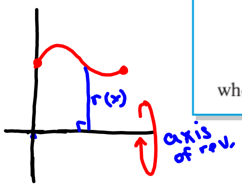
$$S = \ln\left|\sec\tfrac{\pi}{4} + \tan\tfrac{\pi}{4}\right| - \ln|\sec 0 + \tan 0|$$

$$\ln\left|\tfrac{2}{\sqrt{2}} + 1\right| - \underset{0}{\ln}|1 + 0|$$

$$\boxed{\ln|\sqrt{2}+1|} \cancel{- \ln|1|}$$

$$= 0.881$$





DEFINITION OF THE AREA OF A SURFACE OF REVOLUTION

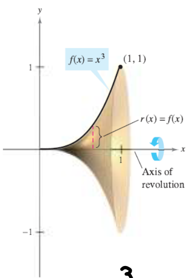
Let $y = f(x)$ have a continuous derivative on the interval $[a, b]$. The area S of the surface of revolution formed by revolving the graph of f about a horizontal or vertical axis is

$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\, dx \qquad y \text{ is a function of } x.$$

where $r(x)$ is the distance between the graph of f and the axis of revolution. If $x = g(y)$ on the interval $[c, d]$, then the surface area is

$$S = 2\pi \int_c^d r(y)\sqrt{1 + [g'(y)]^2}\, dy \qquad x \text{ is a function of } y.$$

where $r(y)$ is the distance between the graph of g and the axis of revolution.



Find the area of the surface formed by revolving the graph of

$$f(x) = x^3$$

on the interval $[0, 1]$ about the x-axis, as shown in Figure 7.46.

$r(x) = x^3$
$f(x) = x^3$
$f'(x) = 3x^2$

$u = 1 + 9x^4$
$du = 36x^3\,dx$

$$SA = 2\pi \int_0^1 r(x)\sqrt{1 + [f'(x)]^2}\, dx$$

$$SA = 2\pi \int_0^1 x^3\sqrt{1 + (3x^2)^2}\, dx$$

$$= \frac{2\pi}{36}\int_0^1 \cancel{3}36x^3(1 + 9x^4)^{1/2}\, dx$$

$$= \frac{\pi}{18}\int_0^1 u^{1/2}\, du$$

$$= \frac{\pi}{18}\left[\frac{2}{3}u^{3/2}\right]_0^1$$

$$= \frac{\cancel{2}}{3}\left(\frac{\pi}{\cancel{18}\,9}\right)\left[(1 + 9x^4)^{3/2}\right]_0^1$$

$$\frac{\pi}{27}\left[10^{3/2} - 1\right] \approx 3.563$$

Homework will be in the Problem book:
page 138 # 1153, 1154, 1158

Find the exact length of the given curve.

1153. $y = x^{3/2}$ from $x = 0$ to $x = 3$

1154. $y = \frac{2}{3}(x+3)^{3/2}$ from $x = 1$ to $x = 6$

Sketch the region R bounded by the given curves, lines, and the x-axis. Then find the volume of the solid generated by revolving R around the x-axis.

1158. $f(x) = \sqrt{x-2},\ x = 3,\ x = 4$