Improper Integrals (BC only)

**DEFINITION OF IMPROPER INTEGRALS WITH INFINITE INTEGRATION LIMITS**

**1.** If $f$ is continuous on the interval $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

**2.** If $f$ is continuous on the interval $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

**3.** If $f$ is continuous on the interval $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$$

where $c$ is any real number (see Exercise 120).

In the first two cases, the improper integral **converges** if the limit exists—otherwise, the improper integral **diverges.** In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverges.

ex. evaluate: $\displaystyle\int_0^{\infty} \frac{dx}{1+x^2}$ replace $\infty$ with a

$$= \int_0^a \frac{1}{1+x^2}\,dx = \tan^{-1}x\Big]_0^a$$

$$= \tan^{-1}a - \cancel{\tan^{-1}0}^{\,0}$$

gotten from graph

Once done integrating

$$= \lim_{a\to\infty} \tan^{-1}a = \tan^{-1}\infty = \frac{\pi}{2}$$

Set = $\lim\limits_{a\to}$ (original number replaced with a)

$4 = a$ because 4 would make denominator undefined

ex. evaluate: $\int_0^4 \sqrt{\frac{4+x}{4-x}}\,dx$

$= \int_0^a \sqrt{\frac{4+x}{4-x}}\,dx \cdot \sqrt{\frac{4+x}{4+x}}$

$= \int_0^a \frac{4+x}{\sqrt{16-x^2}}\,dx = \int_0^a \frac{4}{\sqrt{16-x^2}}\,dx + -\frac{1}{2}\int_0^a \frac{-2x}{\sqrt{16-x^2}}\,dx$ $u = 16-x^2$ $du = -2x\,dx$

$= \int_0^a \frac{4}{\sqrt{16\left(1-\frac{x^2}{16}\right)}}\,dx + -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$

factor out 16

$= 4\int_0^a \frac{\cancel{4}}{\cancel{4}\sqrt{1-\left(\frac{x}{4}\right)^2}}\frac{1}{4}\,dx$ $u = \frac{x}{4}$ $du = \frac{1}{4}\,dx$ $-\frac{1}{2}\int u^{-1/2}\,du$

now in form of $\sin^{-1}$

$= 4\int_0^a \frac{1}{\sqrt{1-u^2}}\,du - \frac{1}{\frac{2}{1/2}} u^{1/2}\Big]_0^a$

$= 4\sin^{-1}\left(\frac{x}{4}\right)\Big]_0^a - \sqrt{16-x^2}\Big]_0^a$

$= \left[4\sin^{-1}\left(\frac{a}{4}\right) - 4\cancel{\sin^{-1}\left(\frac{0}{4}\right)}^{0}\right] - \left[\sqrt{16-a^2} - \cancel{\sqrt{16-0}}^{4}\right]$

$\lim_{a\to 4}\left[4\sin^{-1}\left(\frac{\cancel{a}^{4}}{4}\right) - \sqrt{16-\cancel{a}^{4\,2}} + 4\right.$

$4\left(\frac{\pi}{2}\right) - 0 + 4 = \boxed{2\pi + 4}$

ex. evaluate: $\int_1^\infty \frac{dx}{x} = \int_1^a \frac{1}{x}\,dx = \ln|x|\Big]_1^a$

$= \ln|a| - \cancel{\ln|1|}^{0}$

$\lim_{a\to\infty} \ln a = \infty$

this is said to diverge because $\infty$ is not an actual #

ex. evaluate: $\int_0^{1} \frac{dx}{\sqrt{1-x^2}} =$ change to a because 1 will make integral undefined

$$\int_0^a \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x \Big]_0^a$$

$$= \sin^{-1}a - \cancel{\sin^{-1}0}^{\,0}$$

$$= \lim_{a\to 1} \sin^{-1}a = \sin^{-1}1 = \frac{\pi}{2}$$

converges because $\pi/2$ is a #

assignment: Princeton book: p.412 1-5 all