AP Calculus BC: Partial Fractions Decomposition (use with Intergals)

Lets do the algebra: partial fractions decomposition

If you wanted to add the expression

$$\frac{4}{x+3}+\frac{3}{x-2} = \frac{4(x-2)+3(x+3)}{(x+3)(x-2)}$$

$$= \frac{4x-8+3x+9}{(x+3)(x-2)}$$

$$= \frac{7x+1}{(x+3)(x-2)}$$

But what if you have the answer and want to find the original expression?

Another way of asking this is: What constants A and B exist such that:

$$(x+3)(x-2)\left[\frac{A}{x+3}+\frac{B}{x-2}=\frac{7x+1}{(x+3)(x-2)}\right]$$

Need to do some algebra to solve for A and B

$$A(x-2)+B(x+3)=7x+1$$

$$Ax-2A+Bx+3B=7x+1$$

$$[Ax+Bx]+[-2A+3B]=7x+1$$

$$[A+B]x+[-2A+3B]=7x+1$$

$$2(A+B=7) \quad 2A+2B=14$$

$$-2A+3B=1$$

$$5B=15$$

$$B=3$$

$$A+3=7$$

$$A=4$$

$$\frac{4}{x+3}+\frac{3}{x-2}$$

Why would you need to know this with integrals?

$$\int \frac{7x+1}{(x+3)(x-2)}dx = \int \frac{4}{x+3} + \frac{3}{x-2}\,dx$$

$$= \int \frac{4}{x+3}\,dx + \int \frac{3}{x-2}\,dx$$

$$= 4\int \frac{1}{x+3}\,dx + 3\int \frac{1}{x-2}\,dx$$

$$= 4\ln|x+3| + 3\ln|x-2| + C$$

Another partial fraction: $\frac{2x+4}{(x-1)^2} = \left[\frac{A(x-1)}{(x-1)} + \frac{B(1)}{(x-1)^2}\right](x-1)^2$

$$2x+4 = Ax - A + B$$

$$2 = A \qquad -A + B = 4$$
$$-2 + B = 4$$
$$B = 6$$

$$\frac{2}{x-1} + \frac{6}{(x-1)^2}$$

Now do: $\int \frac{2x+4}{(x-1)^2}dx = \int \frac{2}{x-1} + \frac{6}{(x-1)^2}\,dx$

$$= 2\int \frac{1}{x-1}\,dx + 6\int \frac{1}{(x-1)^2}\,dx \rightarrow \int (x-1)^{-2} = -(x-1)^{-1} = \frac{-1}{x-1}$$

$$= 2\ln|x-1| - \frac{6}{x-1} + C$$

Here is another one: $\int \frac{3x+5}{(x^2+1)(x+2)}dx$

Just Partial fraction

$$\left[\frac{3x+5}{(x^2+1)(x+2)} = \frac{Ax+B}{(x^2+1)} + \frac{C}{x+2}\right](x^2+1)(x+2)$$

$$3x+5 = (Ax+B)(x+2) + C(x^2+1)$$

$$3x+5 = Ax^2 + 2Ax + Bx + 2B + Cx^2 + C$$

$$0x^2 + 3x + 5 = (A+C)x^2 + (2A+B)x + (2B+C)$$

$A+C=0$

$2A+B=3 \Rightarrow B=3-2A$

$2B+C=5 \Rightarrow 2(3-2A)+C=5$

$6-4A+C=5$

$-4A+\cancel{C}=-1$

$-A+\cancel{C}=0$

$-5A=-1$

$A=\frac{1}{5}$

$\frac{1}{5}+C=0$

$C=-\frac{1}{5}$

$B=3-2\left(\frac{1}{5}\right)$

$3-\frac{2}{5}$

$\frac{15}{5}-\frac{2}{5}=\frac{13}{5}$

$$\frac{\frac{1}{5}x+\frac{13}{5}}{x^2+1} + \frac{-\frac{1}{5}}{x+2}$$

$$\boxed{\frac{x+13}{5(x^2+1)} - \frac{1}{5(x+2)}}$$

$$\int \frac{x+13}{5(x^2+1)}dx + \int \frac{-1}{5(x+2)}dx$$

$$\frac{1}{5}\int \frac{x+13}{x^2+1}dx - \frac{1}{5}\int \frac{1}{x+2}dx$$

$$\frac{1}{5}\left[\frac{1}{2}\int \frac{2x}{x^2+1}dx + 13\int \frac{1}{x^2+1}dx\right] - \frac{1}{5}\int \frac{1}{x+2}dx$$

$u=x^2+1$
$du=2x\,dx$

$$\frac{1}{5}\left[\frac{1}{2}\int \frac{1}{u}du + 13\int \frac{1}{x^2+1}dx\right] - \frac{1}{5}\int \frac{1}{x+2}dx$$

$a=1 \quad u=x$

$$\frac{1}{5}\left[\frac{1}{2}\ln|x^2+1| + 13\arctan x\right] - \frac{1}{5}\ln|x+2| + C$$

$$\frac{1}{10}\ln|x^2+1| + \frac{13}{5}\arctan x - \frac{1}{5}\ln|x+2| + C$$

Assignment: princeton book p.407 1-6 all

Find the Partial Fraction Decomposition

1. $\dfrac{x+22}{(x+4)(x-2)}$

2. $\dfrac{x-3}{x(x-3)}$

3. $\dfrac{x^2-7}{x(x^2-4)}$

4. $\dfrac{3x^2+2x+2}{(x^2+1)^2}$

3. $\dfrac{x^2-7}{x(x^2-4)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2-4}$

$x^2-7 = A(x^2-4) + x(Bx+C)$

$x^2-7 = Ax^2 - 4A + Bx^2 + Cx$

$1x^2 - 7 = (A+B)x^2 + Cx - 4A$

$A+B=1$

$C=0$

$-4A=-7$

$A=\frac{7}{4}$

$\frac{7}{4}+B=1$

$B=\frac{-3}{4}$

$\dfrac{\frac{7}{4}}{x} + \dfrac{-\frac{3}{4}x+0}{x^2-4}$

$\boxed{\dfrac{7}{4x} - \dfrac{3x}{4(x^2-4)}}$

4. $\dfrac{3x^2+2x+2}{(x^2+1)^2} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{(x^2+1)^2}$

$(x^2+1)(x^2+1)^2$

$3x^2+2x+2 = (Ax+B)(x^2+1) + Cx + D$

$0x^3 + 3x^2 + 2x + 2 = Ax^3 + Ax + Bx^2 + B + Cx + D$

$0 = A$

$3 = B$

$2 = A+C \Rightarrow C=2$

$2 = D+B \Rightarrow D=-1$

$\dfrac{\cancel{0x}+3}{x^2+1} + \dfrac{2x-1}{(x^2+1)^2}$