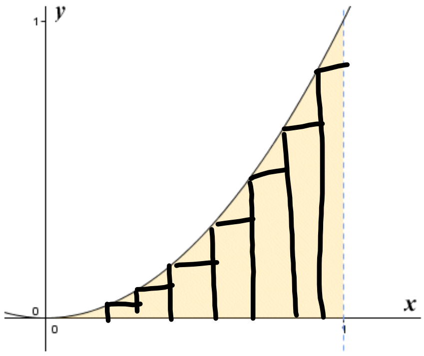
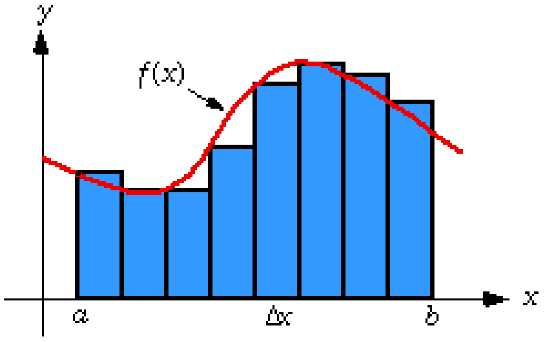
AP Calculus Sec 4.3 Riemann Sums & the Definite Integral

Introducing the Area Under a Curve



A german mathematician, Georg Riemann (1826 -1866) came up with the following formula to find the area under a curve.



Riemann's Sum

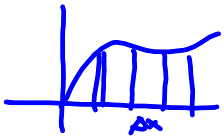
$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

where c_i is the right endpoint of the partition given by $x_1 = \dfrac{i^2}{n^2}$

also where $\Delta x = \dfrac{b-a}{n}$

The width of the largest sub-interval is the norm $\|\Delta\|$

If every subinterval is equal, the partition is regular and denote norm by $\|\Delta\| = \Delta x = \dfrac{b-a}{n}$



This became the definite integral: $\displaystyle\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\,dx.$

where ***a*** is the lower limit of the integration

and ***b*** is the upper limit of the integration

ex. Evaluate the definite integral.

$$\int_{a=-2}^{b=1} 2x\,dx = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

$$c_i = a + i(\Delta x) \qquad \left(\frac{b-a}{n}\right)$$

$$c_i = -2 + i\left(\frac{1--2}{n}\right)$$

$$c_i = -2 + \underbrace{\boxed{\frac{3}{n}}}_{\Delta x} i$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} 2\underbrace{\left(-2+\frac{3}{n}i\right)}_{c_i}\underbrace{\left(\frac{3}{n}\right)}_{\Delta x}$$

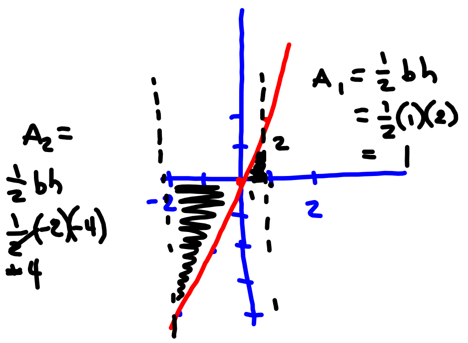
$$= \lim_{n\to\infty} 2\left(\frac{3}{n}\right)\sum_{i=1}^{n}\left(-2+\frac{3}{n}i\right)$$

$$= \lim_{n\to\infty}\frac{6}{n}\left[\sum_{i=1}^{n} -2 + \sum_{i=1}^{n}\frac{3}{n}i\right]$$

$$= \lim_{n\to\infty}\frac{6}{n}\left[\overset{-4n}{\frac{-2n(2)}{2}} + \frac{3}{\cancel{n}}\left(\frac{\cancel{n}(n+1)}{2}\right)\right]$$

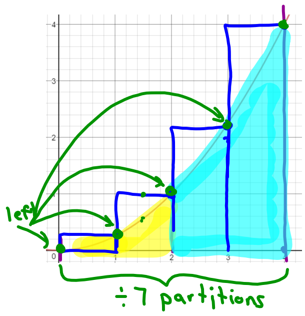
$$= \lim_{n\to\infty}\frac{\overset{3}{\cancel{6}}}{n}\left[\frac{-n+3}{\cancel{2}}\right]$$

$$\lim_{n\to\infty}\frac{-3n+9}{n} = -3$$



Another look at Riemann's Sum

$\int_0^4 \frac{1}{4}x^2\,dx$



$y=\frac{1}{4}x^2$

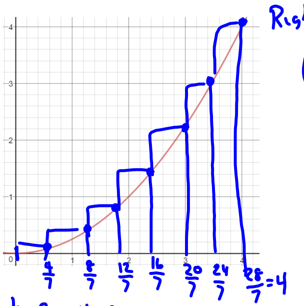
$(2)(1) + (2)(4) = 2+8 = 10$

$(1)(\frac{1}{4}) + (1)(1) + (1)(\frac{9}{4}) + (1)(4) =$

right $\frac{1}{4} + 1 + \frac{9}{4} + 4 = \frac{10}{4} + 5 = \frac{30}{4} = \frac{15}{2}$

left $(1)(0) + (1)(\frac{1}{4}) + (1)(1) + (1)(\frac{9}{4}) =$

$0 + \frac{1}{4} + 1 + \frac{9}{4} = \frac{14}{4} = \frac{7}{2}$



$\frac{b-a}{n} = \frac{4-0}{7} = \frac{4}{7}$

Right hand

$(\frac{4}{7})(.08) + (\frac{4}{7})(.32) + (\frac{4}{7})(.73) + (\frac{4}{7})(1.3) +$

$(\frac{4}{7})(2.04) + (\frac{4}{7})(2.9) + (\frac{4}{7})(4) =$

To make this easier, factor out 4/7

$\frac{4}{7}(.08 + .32 + .73 + 1.3 + 2.04 + 2.9 + 4) = \boxed{6.497}$

left right

x	0	$\frac{4}{7}$	$\frac{8}{7}$	$\frac{12}{7}$	$\frac{16}{7}$	$\frac{20}{7}$	$\frac{24}{7}$	4
y	0	0.08	.32	.73	1.3	2.04	2.9	4

Assignment: PR p.313 #1, 2, 11

Problem book p.106 # 920-923

From this example is the area formula:

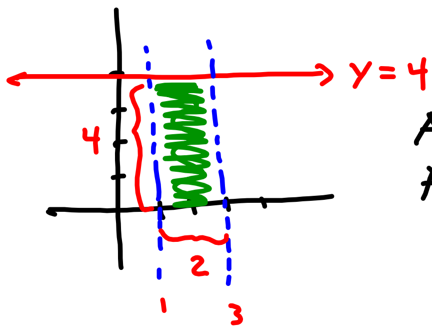
$$area = \int_a^b f(x)\,dx$$

ex. $\int_1^3 4\,dx$ $\quad$ y = 4

$4x\Big]_1^3 = 4(3) - 4(1)$

$12 - 4$

8



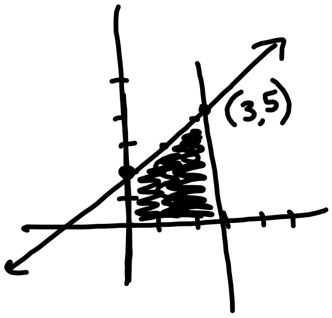
A = bh

A = 2·4 = 8

ex. $\displaystyle\int_0^3 (x+2)\,dx \longrightarrow \left.\frac{x^2}{2} + 2x \right]_0^3$

$y = x + 2$

$\left(\frac{9}{2} + 6\right) - (0) = \frac{21}{2}$



$A = \frac{1}{2}h(b_1 + b_2)$

$A = \frac{1}{2}\,3(2 + \cancel{4}\,5)$

$A = \frac{3}{2}(7) = \frac{21}{2}$

11. $-\frac{1}{2}\displaystyle\int -2x\, 4^{-x^2}\,dx$

$u = -x^2$

$du = -2x\,dx$

$-\frac{1}{2}\displaystyle\int 4^u\,du = -\frac{1}{2}\left(\frac{1}{\ln 4}\right)4^u + C$

$= -\frac{1}{2\ln 4}\left(4^{-x^2}\right) + C$

$= \frac{-1}{4^{x^2}\ln 16} + C$ or $\frac{-4^{-x^2}}{\ln 16} + C$

12. $\int 7^{\sin x} \cos x \, dx$ $\qquad u = \sin x$

$du = \cos x \, dx$

$\int 7^{u} \, du$

$= \frac{1}{\ln 7} 7^{u} + C$

$= \frac{7^{\sin x}}{\ln 7} + C$