Integration Review for AP Calculus AB/BC Exam

This review contains all the definitions and rules with examples.

Summary

Indefinite integral problems come in many different types on the AP Calculus Exams. Remember that an indefinite integral is the most general antiderivative of a function.

Among the wide range of techniques available, most problems can be handled by one or more of the following methods.

- Basic antiderivative formulas, including the Power Rule and rules for special kinds of functions (such as trigonometric and exponential).
- Substitution
- Integration by parts
- Partial fractions

After much practice, you will be able to choose the best technique for each integral problem. Just like a good carpenter, using the right tool makes the job easy. You may even come to enjoy the challenge of indefinite integrals!

Integrals

Definitions

Definite Integral: Suppose $f(x)$ is continuous on $[a,b]$. Divide $[a,b]$ into n subintervals of width Δx and choose x_i^* from each interval.

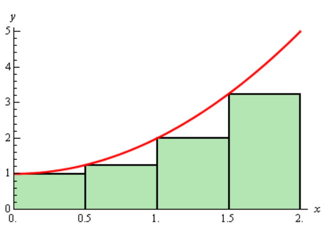
Then $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{\infty} f(x_i^*)\Delta x$.

Anti-Derivative : An anti-derivative of $f(x)$ is a function, $F(x)$, such that $F'(x) = f(x)$.

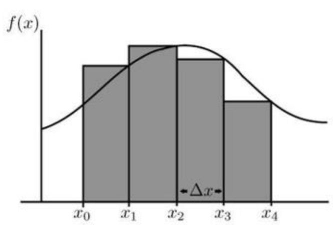
Indefinite Integral : $\int f(x)dx = F(x) + c$ where $F(x)$ is an anti-derivative of $f(x)$.

Riemann's Sum

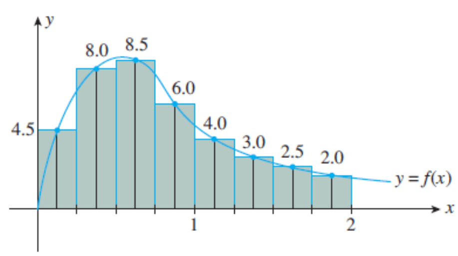
Left-hand



Right-hand



Midpoint

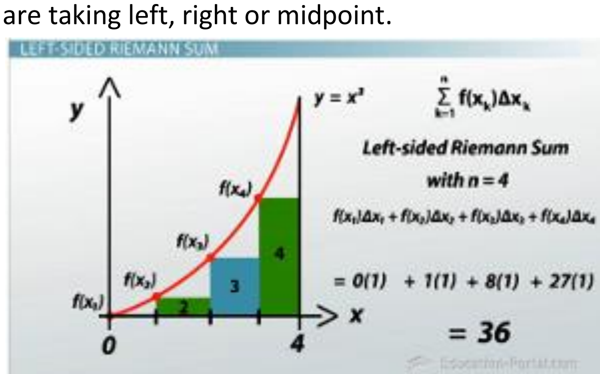


Very simplified formula for Riemann's Sum:

$$A = [\ (h_1 + h_2 + \ldots + h_n)]$$

Where Δx the width of the rectangles and h_1 is the height of the rectangles. The h would change depending on whether you are taking left, right or midpoint.

Example:



Fundamental Theorem of Calculus

Part I : If $f(x)$ is continuous on $[a,b]$ then $g(x)=\int_a^x f(t)\,dt$ is also continuous on $[a,b]$ and $g'(x)=\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

Part II : $f(x)$ is continuous on $[a,b]$, $F(x)$ is an anti-derivative of $f(x)$ (*i.e.* $F(x)=\int f(x)\,dx$) then $\int_a^b f(x)\,dx = F(b)-F(a)$.

Variants of Part I :

$$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = u'(x)\,f\left[u(x)\right]$$

$$\frac{d}{dx}\int_{v(x)}^{b} f(t)\,dt = -v'(x)\,f\left[v(x)\right]$$

$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\,dt = u'(x)\,f\left[u(x)\right]-v'(x)\,f\left[v(x)\right]$$

Common Integrals

$\int k\,dx = k\,x + c$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1}+c,\ n\neq -1$

$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x|+c$

$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b|+c$

$\int \ln u\,du = u\ln(u)-u+c$

$\int \mathbf{e}^u\,du = \mathbf{e}^u + c$

$\int \cos u\,du = \sin u + c$

$\int \sin u\,du = -\cos u + c$

$\int \sec^2 u\,du = \tan u + c$

$\int \sec u \tan u\,du = \sec u + c$

$\int \csc u \cot u\,du = -\csc u + c$

$\int \csc^2 u\,du = -\cot u + c$

$\int \tan u\,du = \ln|\sec u| + c$

$\int \sec u\,du = \ln|\sec u + \tan u| + c$

$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+c$

$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right)+c$

Standard Integration Techniques

Note that at many schools all but the Substitution Rule tend to be taught in a Calculus II class.

***u* Substitution :** The substitution $u=g(x)$ will convert $\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$ using $du = g'(x)\,dx$. For indefinite integrals drop the limits of integration.

Ex. $\int_1^2 5x^2\cos(x^3)\,dx$

$u=x^3 \ \Rightarrow\ du = 3x^2dx \ \Rightarrow\ x^2dx = \frac{1}{3}du$

$x=1 \ \Rightarrow\ u=1^3=1 \ :: \ x=2 \ \Rightarrow\ u=2^3=8$

$\int_1^2 5x^2\cos(x^3)\,dx = \int_1^8 \frac{5}{3}\cos(u)\,du$

$= \frac{5}{3}\sin(u)\Big|_1^8 = \frac{5}{3}\left(\sin(8)-\sin(1)\right)$

Integration by Parts : $\int u\,dv = uv - \int v\,du$ and $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$. Choose u and dv from integral and compute du by differentiating u and compute v using $v=\int dv$.

Ex. $\int x\mathbf{e}^{-x}\,dx$

$u=x \quad dv=\mathbf{e}^{-x} \quad \Rightarrow \quad du=dx \quad v=-\mathbf{e}^{-x}$

$\int x\mathbf{e}^{-x}\,dx = -x\mathbf{e}^{-x}+\int \mathbf{e}^{-x}\,dx = -x\mathbf{e}^{-x}-\mathbf{e}^{-x}+c$

Ex. $\int_3^5 \ln x\,dx$

$u=\ln x \quad dv=dx \quad \Rightarrow \quad du=\frac{1}{x}dx \quad v=x$

$\int_3^5 \ln x\,dx = x\ln x\Big|_3^5 - \int_3^5 dx = \left(x\ln(x)-x\right)\Big|_3^5$

$= 5\ln(5)-3\ln(3)-2$

Trig Substitutions : If the integral contains the following root use the given substitution and formula to convert into an integral involving trig functions.

$\sqrt{a^2-b^2x^2} \Rightarrow x=\frac{a}{b}\sin\theta$	$\sqrt{b^2x^2-a^2} \Rightarrow x=\frac{a}{b}\sec\theta$	$\sqrt{a^2+b^2x^2} \Rightarrow x=\frac{a}{b}\tan\theta$
$\cos^2\theta = 1-\sin^2\theta$	$\tan^2\theta=\sec^2\theta-1$	$\sec^2\theta=1+\tan^2\theta$

Ex. $\int \frac{16}{x^2\sqrt{4-9x^2}}\,dx$

$x=\frac{2}{3}\sin\theta \Rightarrow dx=\frac{2}{3}\cos\theta\,d\theta$

$\sqrt{4-9x^2}=\sqrt{4-4\sin^2\theta}=\sqrt{4\cos^2\theta}=2|\cos\theta|$

Recall $\sqrt{x^2}=|x|$. Because we have an indefinite integral we'll assume positive and drop absolute value bars. If we had a definite integral we'd need to compute θ's and remove absolute value bars based on that and,

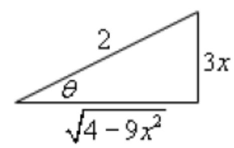
$$|x|=\begin{cases} x & \text{if } x\ge 0 \\ -x & \text{if } x<0 \end{cases}$$

In this case we have $\sqrt{4-9x^2}=2\cos\theta$.

$$\int \frac{16}{\frac{4}{9}\sin^2\theta(2\cos\theta)}\left(\tfrac{2}{3}\cos\theta\right)d\theta = \int \frac{12}{\sin^2\theta}\,d\theta$$

$$=\int 12\csc^2\,d\theta = -12\cot\theta + c$$

Use Right Triangle Trig to go back to x's. From substitution we have $\sin\theta=\frac{3x}{2}$ so,



From this we see that $\cot\theta=\frac{\sqrt{4-9x^2}}{3x}$. So,

$$\int \frac{16}{x^2\sqrt{4-9x^2}}\,dx = -\frac{4\sqrt{4-9x^2}}{x}+c$$

Partial Fractions : If integrating $\int \frac{P(x)}{Q(x)}\,dx$ where the degree of $P(x)$ is smaller than the degree of $Q(x)$. Factor denominator as completely as possible and find the partial fraction decomposition of the rational expression. Integrate the partial fraction decomposition (P.F.D.). For each factor in the denominator we get term(s) in the decomposition according to the following table.

Factor in $Q(x)$	Term in P.F.D	Factor in $Q(x)$	Term in P.F.D
$ax+b$	$\frac{A}{ax+b}$	$(ax+b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$	$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Ex. $\int \frac{7x^2+13x}{(x-1)(x^2+4)}\,dx$

$$\int \frac{7x^2+13x}{(x-1)(x^2+4)}\,dx = \int \frac{4}{x-1}+\frac{3x+16}{x^2+4}\,dx$$

$$= \int \frac{4}{x-1}+\frac{3x}{x^2+4}+\frac{16}{x^2+4}\,dx$$

$$= 4\ln|x-1|+\tfrac{3}{2}\ln\left(x^2+4\right)+8\tan^{-1}\left(\tfrac{x}{2}\right)$$

Here is partial fraction form and recombined.

$$\frac{7x^2+13x}{(x-1)(x^2+4)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+4}=\frac{A(x^2+4)+(Bx+C)(x-1)}{(x-1)(x^2+4)}$$

Set numerators equal and collect like terms.

$$7x^2+13x=(A+B)x^2+(C-B)x+4A-C$$

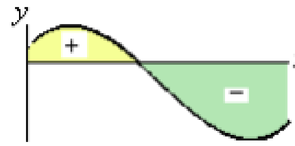
Set coefficients equal to get a system and solve to get constants.

$A+B=7 \qquad C-B=13 \qquad 4A-C=0$

$A=4 \qquad B=3 \qquad C=16$

Applications of Integrals

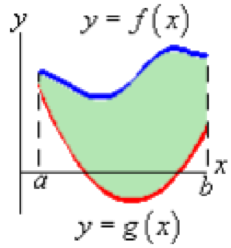
Net Area : $\int_a^b f(x)\,dx$ represents the net area between $f(x)$ and the x-axis with area above x-axis positive and area below x-axis negative.



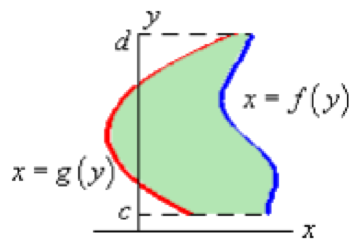
Area Between Curves : The general formulas for the two main cases for each are,

$$y = f(x) \Rightarrow A = \int_a^b [\text{upper function}] - [\text{lower function}]\,dx \quad \& \quad x = f(y) \Rightarrow A = \int_c^d [\text{right function}] - [\text{left function}]\,dy$$

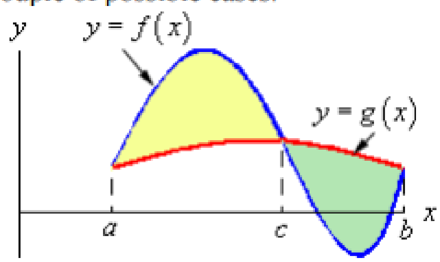
If the curves intersect then the area of each portion must be found individually. Here are some sketches of a couple possible situations and formulas for a couple of possible cases.



$$A = \int_a^b f(x) - g(x)\,dx$$



$$A = \int_c^d f(y) - g(y)\,dy$$

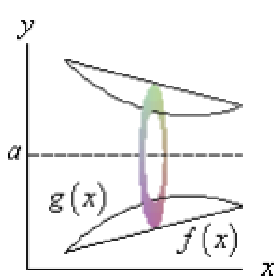


$$A = \int_a^c f(x) - g(x)\,dx + \int_c^b g(x) - f(x)\,dx$$

Volumes of Revolution : The two main formulas are $V = \int A(x)\,dx$ and $V = \int A(y)\,dy$. Here is some general information about each method of computing and some examples.

Rings	**Cylinders**
$A = \pi\left((\text{outer radius})^2 - (\text{inner radius})^2\right)$	$A = 2\pi\,(\text{radius})\,(\text{width / height})$
Limits: x/y of right/bot ring to x/y of left/top ring	Limits : x/y of inner cyl. to x/y of outer cyl.
Horz. Axis use $f(x)$, $g(x)$, $A(x)$ and dx. Vert. Axis use $f(y)$, $g(y)$, $A(y)$ and dy.	Horz. Axis use $f(y)$, $g(y)$, $A(y)$ and dy. Vert. Axis use $f(x)$, $g(x)$, $A(x)$ and dx.

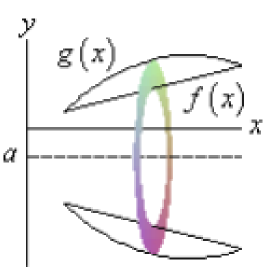
Ex. Axis : $y = a > 0$



outer radius : $a - f(x)$

inner radius : $a - g(x)$

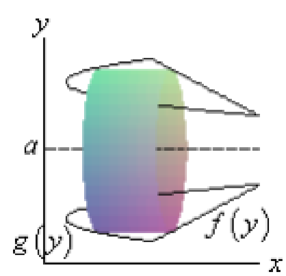
Ex. Axis : $y = a \le 0$



outer radius: $|a| + g(x)$

inner radius: $|a| + f(x)$

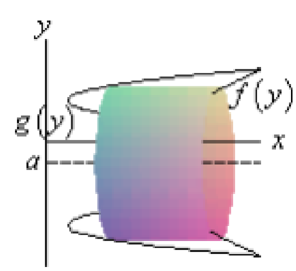
Ex. Axis : $y = a > 0$



radius : $a - y$

width : $f(y) - g(y)$

Ex. Axis : $y = a \le 0$



radius : $|a| + y$

width : $f(y) - g(y)$

These are only a few cases for horizontal axis of rotation. If axis of rotation is the x-axis use the $y = a \le 0$ case with $a = 0$. For vertical axis of rotation ($x = a > 0$ and $x = a \le 0$) interchange x and y to get appropriate formulas.

Work : If a force of $F(x)$ moves an object in $a \le x \le b$, the work done is $W = \int_a^b F(x)\,dx$

Average Function Value : The average value of $f(x)$ on $a \le x \le b$ is $f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$

Arc Length Surface Area : Note that this is often a Calc II topic. The three basic formulas are,

$L = \int_a^b ds$ $\qquad SA = \int_a^b 2\pi y\,ds$ (rotate about *x*-axis) $\qquad SA = \int_a^b 2\pi x\,ds$ (rotate about *y*-axis)

where *ds* is dependent upon the form of the function being worked with as follows.

$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ if $y = f(x),\ a \le x \le b$ $\qquad ds = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ if $x = f(t), y = g(t),\ a \le t \le b$

$ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ if $x = f(y),\ a \le y \le b$ $\qquad ds = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$ if $r = f(\theta),\ a \le \theta \le b$

With surface area you *may* have to substitute in for the *x* or *y* depending on your choice of *ds* to match the differential in the *ds*. With parametric and polar you will always need to substitute.

Improper Integral

An improper integral is an integral with one or more infinite limits and/or discontinuous integrands. Integral is called convergent if the limit exists and has a finite value and divergent if the limit doesn't exist or has infinite value. This is typically a Calc II topic.

Infinite Limit

1. $\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$
2. $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$
3. $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$ provided BOTH integrals are convergent.

Discontinuous Integrand

1. Discont. at a: $\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$
2. Discont. at b : $\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$
3. Discontinuity at $a < c < b$: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ provided both are convergent.

Comparison Test for Improper Integrals : If $f(x) \ge g(x) \ge 0$ on $[a,\infty)$ then,

1. If $\int_a^\infty f(x)\,dx$ conv. then $\int_a^\infty g(x)\,dx$ conv.
2. If $\int_a^\infty g(x)\,dx$ divg. then $\int_a^\infty f(x)\,dx$ divg.

Useful fact : If $a > 0$ then $\int_a^\infty \frac{1}{x^p}\,dx$ converges if $p > 1$ and diverges for $p \le 1$.

Approximating Definite Integrals

For given integral $\int_a^b f(x)\,dx$ and a *n* (must be even for Simpson's Rule) define $\Delta x = \frac{b-a}{n}$ and divide $[a,b]$ into *n* subintervals $[x_0,x_1]$, $[x_1,x_2]$, … , $[x_{n-1},x_n]$ with $x_0 = a$ and $x_n = b$ then,

Midpoint Rule : $\int_a^b f(x)\,dx \approx \Delta x\left[f(x_1^*)+f(x_2^*)+\cdots+f(x_n^*)\right]$, x_i^* is midpoint $[x_{i-1},x_i]$

Trapezoid Rule : $\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)++2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\right]$

Simpson's Rule : $\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]$