AP Calculus—Integration Practice

I. **Integration by substititition.**

Basic Idea: If $u = f(x)$, then $du = f'(x)dx$.

Example. We have

$$\int \frac{x\,dx}{x^4+1} \overset{u=x^2}{\underset{dx=2x\,dx}{=}} \frac{1}{2}\int \frac{du}{u^2+1} = \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\tan^{-1} x^2 + C$$

Practice Problems:

1. $\displaystyle\int x^3\sqrt{4+x^4}\,dx$

2. $\displaystyle\int \frac{dx}{x\ln x}$

3. $\displaystyle\int \frac{(x+5)\,dx}{\sqrt{x+4}}$

4. In each integral below, find the integer n that allows for an integration by **substitution**. Then perform the integration.
   (a) $\displaystyle\int x^n\sqrt{1-x^4}\,dx$
   (b) $\displaystyle\int \frac{x^n}{\sqrt{1-x^4}}\,dx$ (there are two very natural choices for n).
   (c) $\displaystyle\int \frac{x^n}{1+x^{10}}\,dx$ (there are two very natural choices for n).
   (d) $\displaystyle\int \frac{x^6}{1+x^n}\,dx$
   (e) $\displaystyle\int x^n e^{-x^2}\,dx$
   (f) $\displaystyle\int x^n e^{2x^5}\,dx$
   (g) $\displaystyle\int x^5\sqrt{1-x^n}\,dx$

(h) $\displaystyle\int \frac{x^6}{\sqrt{1-x^n}}\,dx$

(i) $\displaystyle\int \frac{dx}{x^n \ln x}$

(j) $\displaystyle\int \frac{dx}{x^n(\ln x)^7}$

(k) $\displaystyle\int x^n \sin(x^6)\,dx$

(l) $\displaystyle\int \frac{\sin^n x \cos x}{\sqrt{3+\sin^4 x}}\,dx$

(m) $\displaystyle\int \frac{\sin^3 x \cos x}{\sqrt{3+\sin^n x}}\,dx$

II. **Integration by Parts:**

Basic Idea: $\int u\,dv \;=\; uv - \int v\,du$

(Try to substitute u so that $\dfrac{du}{dx}$ is simpler than u and so that v is no more complicated than dv.)

Example. We have

$$\int x\sin x\,dx \overset{u=x,\quad dv=\sin x\,dx}{\underset{du=dx,\quad v=-\cos x\,dx}{=}} -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x$$

Notice that in the above, setting $u = x$ yields $\dfrac{du}{dx} = 1$ (i.e., $du = dx$), which is **simpler** and $dv = \sin x\,dx$ which gives $v = -\cos x$, which is no more complicated.

Practice Problems:

1. $\displaystyle\int xe^{-x/10}\,dx$

2. $\displaystyle\int x^2e^{-x/10}\,dx.$

3. $\displaystyle\int x^2\ln x\,dx$

4. $\displaystyle\int x^n\ln x\,dx$ (n is an integer)

5. $\int x^2 \sin x \, dx$

6. $\int x^3 e^{-x^2} \, dx$

7. $\int x^3 \sqrt{x^2+1} \, dx$

8. Assume that $\int f(x) \, dx = g(x)$, that $\int g(x) \, dx = h(x)$ and compute

   (a) $\int x^3 f(x^2) \, dx$

   (b) $\int x^{2n-1} f(x^n) \, dx$

9. $\int \sin^{-1} x \, dx$

10. $\int \left(\sin^{-1} x\right)^2 \, dx$

11. $\int \tan^{-1} x \, dx$

12. $\int \sec^3 \theta \, d\theta$ (Hint: write $\sec^3 \theta = \sec \theta (1 + \tan^2 \theta)$ and integrate $\sec \theta \tan^2 \theta$ by parts.)

III. **Trigonometric Substitutions.**

Basic Idea:

$\boldsymbol{a^2 - x^2}$ For expressions like $a^2 - x^2$ substitute $x = a \sin \theta$. Then $x^2 - x^2 = a^2 \cos^2 \theta$ and $dx = a \cos \theta \, d\theta$.

$\boldsymbol{a^2 + x^2}$ For expressions like $a^2 + x^2$ substitute $x = a \tan \theta$. Then $x^2 + x^2 = a^2 \sec^2 \theta$ and $dx = a \sec^2 \theta \, d\theta$.

$\boldsymbol{x^2 - a^2}$ For expressions like $x^2 - a^2$ substitute $x = a \sec \theta$. Then $x^2 - a^2 = \tan^2 \theta$, and $dx = \sec \theta \tan \theta \, d\theta$.

Example 1. We have

$$\int \sqrt{4-x^2}\,dx \overset{x=2\sin\theta}{\underset{dx=2\cos\theta\,d\theta}{=}} 4\int \cos^2\theta\,d\theta$$

$$= 2\int (1+\cos 2\theta)\,d\theta$$

$$= 2\theta + \sin 2\theta + C$$

$$= 2\sin^{-1}\left(\frac{x}{2}\right) + 2\sin\theta\cos\theta + C$$

$$= 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}x\sqrt{4-x^2} + C$$

SECOND EXAMPLE. In many integrations involving a trig substitution, there is the need to integrate $\sec\theta$. This is easy but requires a trick:

$$\int \sec\theta\,d\theta = \int \frac{\sec\theta(\sec\theta+\tan\theta)\,d\theta}{\sec\theta+\tan\theta}$$

$$\overset{u=\sec\theta+\tan\theta}{\underset{du=\sec\theta(\sec\theta+\tan\theta)\,d\theta}{=}} \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln|\sec\theta+\tan\theta| + C$$

In an entirely similar fashion, one shows that $\int \csc\theta\,d\theta = -\ln|\csc\theta+\cot\theta| + C$.

Example 2. Here's one that uses the above ideas.

$$\int \frac{\sqrt{a^2-x^2}\,dx}{x} \overset{x=a\sin\theta}{\underset{dx=a\cos\theta\,d\theta}{=}} a\int \frac{\cos^2\theta\,d\theta}{\sin\theta}$$

$$= a\int \frac{(1-\sin^2\theta)\,d\theta}{\sin\theta}$$

$$= a\int (\csc\theta - \sin\theta)\,d\theta$$

$$= -a\ln|\csc\theta+\cot\theta| + a\cos\theta + C$$

$$= \sqrt{a^2-x^2} - a\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C$$

Practice Problems:

1. $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$

2. $\int \frac{dx}{x\sqrt{1-x^2}}$

3. $\int \frac{dx}{x\sqrt{a^2+x^2}}$

4. $\int \sqrt{4+x^2}\,dx$ (Hint: see problem 12 page 3.)

5. $\int \frac{dx}{a^2-x^2}$ (It might be easier to do this by partial fractions.)

6. $\int \frac{\sqrt{x^2-a^2}}{x}\,dx$

7. $\int \frac{dx}{(a^2+x^2)^2}$

8. $\int \sin^{-1} x\,dx$ (Let $x = \sin\theta$)

9. $\int \left(\sin^{-1} x\right)^2\,dx$

10. $\int \tan^{-1} x\,dx$

IV. **Integration by Partial Fractions.**

Basic Idea: This is used to integrate rational functions. Namely, if $R(x) = \frac{p(x)}{q(x)}$ is a rational function, with $p(x)$ and $q(x)$ polynomials, then we can factor $q(x)$ into a product of linear and irreducible quadratic factors, possibly with multiplicities. For each power $(x-\alpha)^n$ of a linear factor, the expansion of $R(x)$ will contain terms of the form

$$\frac{a_1}{x-\alpha} + \frac{a_2}{(x-\alpha)^2} + \cdots + \frac{a_n}{(x-\alpha)^n},$$

where $a_1,\ a_2,\ \ldots, a_n$ are all real constants. For each power $(x^2+\alpha x+\beta)^m$ of an irreducible quadratic factor, then the expansion of $R(x)$ will contain terms of the form

$$\frac{a_1x+b_1}{x^2+\alpha x+\beta}+\frac{a_2x+b_2}{(x^2+\alpha x+\beta)^2}+\cdots+\frac{a_mx+b_m}{(x^2+\alpha x+\beta)^m},$$

where $a_1,\ a_2,\ \ldots, a_m$ and $b_1,\ b_2,\ \ldots, b_m$ are real constants.

The determination of the constants above is a purely **algebraic** process. For example, in decomposing the rational function $R(x)=\dfrac{x+1}{(x-2)(x^2+4)}$ we set this up as follows:

$$\frac{x+1}{(x-2)(x^2+4)} = \frac{a}{x-2}+\frac{bx+c}{x^2+4}.$$

At this juncture, there are a number of approaches. One is to multiply through, clearing all denominators and equating coefficients in the resulting polynomial equation:

$$x+1 = a(x^2+4)+(bx+c)(x-2).$$

This quickly yields

$$\begin{aligned} a+b &= 0,\\ -2b+c &= 1,\\ 4a-2c &= 1, \end{aligned}$$

from which we conclude that $a=3/8,\ b=-3/8$, and $c=1/4$.

To compute the indefinite integral $\int R(x)\,dx$, we need to be able to compute integrals of the form

$$\int\frac{a}{(x-\alpha)^n}\,dx \quad\text{and}\quad \int\frac{bx+c}{(x^2+\alpha x+\beta)^m}\,dx.$$

Those of the first type above are simple; a substitution $u=x-\alpha$ will serve to finish the job. Those of the second type can, via completing the square, be reduced to integrals of the form $\dfrac{bx+c}{(x^2+a^2)^m}\,dx$. This involves a sum of two integrals: those of the form $\displaystyle\int\frac{bx}{(x^2+a^2)^m}\,dx$ can be computed via the substitution $u=x^2+a^2$; those of the form $\displaystyle\int\frac{c}{(x^2+a^2)^m}\,dx$ can be handled by the appropriate trigonometric substitution (viz., $x=a\tan\theta$).

From the above work, we may now finish our example.

$$\begin{aligned}\int\frac{x+1}{(x-2)(x^2+4)}\,dx &= \frac{3}{8}\int\frac{dx}{x-2}-\frac{1}{8}\int\frac{3x-2}{x^2+4}\,dx\\ &= \frac{3}{8}\ln|x-2|-\frac{3}{16}\ln(x^2+4)+\frac{1}{8}\tan^{-1}\left(\frac{x}{2}\right)+C.\end{aligned}$$

Practice Problems:

1. $\displaystyle\int \frac{5x-3}{x^2-2x-3}\,dx$

2. $\displaystyle\int \frac{6x+7}{(x+2)^2}\,dx$

3. $\displaystyle\int \frac{2x^3-4x^2-x-3}{x^2-x2-3}\,dx$

4. $\displaystyle\int \frac{dx}{x(x^2+1)}$

5. $\displaystyle\int \left(\frac{1}{x^2+1}-\frac{1}{x^2-2x+5}\right)dx$

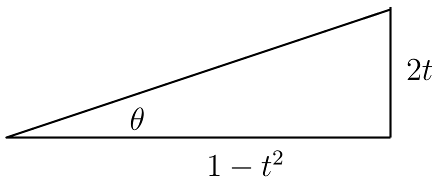
6. $\displaystyle\int \frac{x^3+2x^2+2}{(x^2+1)^2}\,dx$

V. The $t = \tan\frac{1}{2}\theta$ substitution

Basic Idea: This technique is particularly useful in computing definite integrals having integrands of the form $\dfrac{1}{a+b\cos\theta}$ or $\dfrac{1}{a+b\sin\theta}$. If we let $t=\tan\frac{1}{2}\theta$, then using the double-angle identity for

the tangent:

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A},$$



we obtain immediately that

$$\tan\theta = \frac{2t}{1-t^2}.$$

From the picture depicted to the right, we obtain, therefore, that

$$\sin\theta = \frac{2t}{1+t^2} \quad \text{and that} \quad \cos\theta = \frac{1-t^2}{1+t^2}.$$

EXAMPLE. We use the above to compute $\displaystyle\int_0^{\pi/2} \frac{4}{3+5\sin\theta}\,d\theta$.

With the substitution $t=\tan\frac{1}{2}\theta$, we have $\dfrac{dt}{d\theta}=\frac{1}{2}\sec^2\frac{1}{2}\theta=\dfrac{1+t^2}{2}$. From this it follows that $d\theta=\dfrac{2\,dt}{1+t^2}$; we now proceed as follows:

$$\begin{aligned}
\int_0^{\pi/2} \frac{4}{3+5\sin\theta}\,d\theta &\overset{t=\tan\frac{1}{2}\theta}{=} \int_0^1 \frac{4}{3+10t/(1+t^2)} \times \frac{2}{1+t^2}\,dt \\
&= \int_0^1 \frac{8}{3t^2+10t+3}\,dt \\
&= \int_0^1 \left(\frac{3}{3t+1} - \frac{1}{t+3}\right) dt \\
&= \ln(3t+1) - \ln(t+3)\Big|_0^1 \\
&= \ln 3
\end{aligned}$$

Practice Problems:[1]

1. $\displaystyle\int_0^{\pi/2} \frac{3}{1+\sin\theta}\,d\theta$

2. $\displaystyle\int_0^{2\pi/3} \frac{3}{5+4\cos\theta}\,d\theta$

3. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{3}{4+5\cos\theta}\,d\theta$

4. $\displaystyle\int_0^{\pi/2} \frac{5}{3\sin\theta+4\cos\theta}\,d\theta$

VI. **Differential Equations—Variables Separable.**

Basic Idea: The IB syllabus for Calculus (Core Topic 7) contains a component relating to a special class of differential equations, namely those having the variables separable. Specifically, this relates to those differential equations $\frac{dy}{dx} = f(x,y)$, where the function $f(x,y)$ can be written in the form $f(x,y) = g(x)h(y)$, for suitable functions g and h. Such a differential equation can, in principle, yield an implicit solution for y via separating the variables and integrating:

$$\frac{dy}{dx} = g(x)h(y) \Rightarrow \frac{dy}{h(y)} = g(x)\,dx \Rightarrow \int \frac{dy}{h(y)} = \int g(x)\,dx.$$

Assuming that the integrations can be performed (which is a significant assumption!) we arrive at an equation of the type $H(y) = G(x)$, which defines y implicitly as a function of x.

[1] These (and the example above) have been lifted from Sadler and Thorning, pp 500–501:

EXAMPLE 1. Consider the differential equation $\frac{dy}{dx} = -3x^2y$, subject to the initial condition $y(0) = 2$. We proceed as above:

$$\frac{dy}{dx} = -3x^2y \Rightarrow \frac{dy}{y} = -3x^2\,dx \Rightarrow \int \frac{dy}{y} = -\int 3x^2\,dx \Rightarrow \ln|y| = -x^3 + C.$$

The above can be rendered more explicit by exponentiating both sides and setting $K = e^C$ (an arbitrary constant); the result is $y = Ke^{-x^3}$. Finally, use the initial condition $y(0) = 2$: $2 = Ke^0 = K$, and so the resulting solution is $y = 2e^{-x^3}$.

EXAMPLE 2. This time, we consider the so-called **logistic differential equation**

$$\frac{dy}{dx} = ay(1-y), \quad \text{where } a > 0 \quad \text{is a constant, } y(0) = .2.$$

Upon separating the variables, we obtain

$$\int \frac{dy}{y(1-y)} = \int a\,dx.$$

Next, using the partial fraction decomposition $\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}$, we obtain

$$\int \left(\frac{1}{y} + \frac{1}{1-y}\right) dy = \int a\,dx$$

from which it follows that

$$\ln|y| - \ln|1-y| = ax + C \Rightarrow \frac{y}{1-y} = Ke^{ax}.$$

Solving for y in terms of x is fairly easily done; the result is

$$y = \frac{Ke^{ax}}{1+Ke^{ax}} = \frac{1}{1+Be^{-ax}},$$

where $B = K^{-1}$, again, an arbitrary constant.

We conclude with a few words of terminology. What we have considered above are usually called **ordinary differential equations**, typically abbreviated ODE. These are to be distinguished from **partial differential equations**, which, as you can guess, involve partial derivatives and are typically much harder.[2] Next, the arbitrary constant which arises in the integration of an ODE is typically solved via the specification of

[2]One of the "Millennium Problems" is to help the mathematical community arrive at a better understanding of the Navier-Stokes equations, which are expressible through partial differential equations.

an initial condition, often expressed in the form $y(0) = y_0$. If both the differential equation and the initial condition are expressed, say by writing

$$\frac{dy}{dx} = f(x, y), \;\; y(0) = y_0,$$

we call the above an **initial value problem**, or IVP.

Practice Problems: Solve the following IVPs. (Unless it is convenient to do so, do not attempt to write the solution y **explicitly** as a function of x.)

1. $\dfrac{dy}{dx} = xy, \;\; y(0) = 1.$

2. $y\dfrac{dy}{dx} = x^2, \;\; y(0) = 1.$

3. $\dfrac{dy}{dx} = -2x(y+3), \;\; y(0) = 1.$

4. $\dfrac{dy}{dx} = \dfrac{x^2y + y}{x^2 - 1}, \;\; y(0) = 2.$