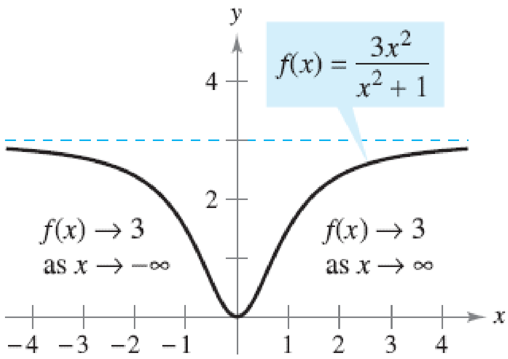
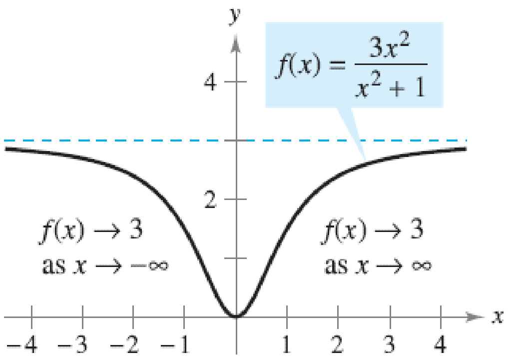
AP Calculus BC Limits at Infinity

Look at the graph of $f(x) = \dfrac{3x^2}{x^2 + 1}$



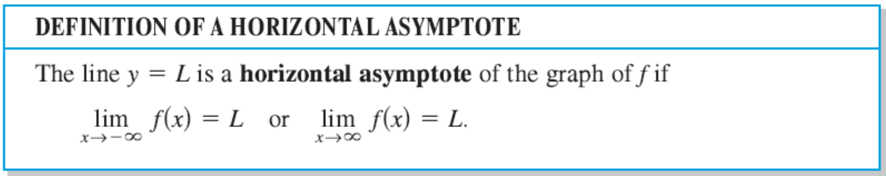
As x goes to -∞, y would approach? 3

As x goes to +∞, y would approach? 3



So going to - ∞ or going to + ∞ results in the same answer.

This becomes the definition of a horizontal asymptote.



**DEFINITION OF A HORIZONTAL ASYMPTOTE**

The line $y = L$ is a **horizontal asymptote** of the graph of $f$ if

$$\lim_{x\to -\infty} f(x) = L \quad \text{or} \quad \lim_{x\to\infty} f(x) = L.$$

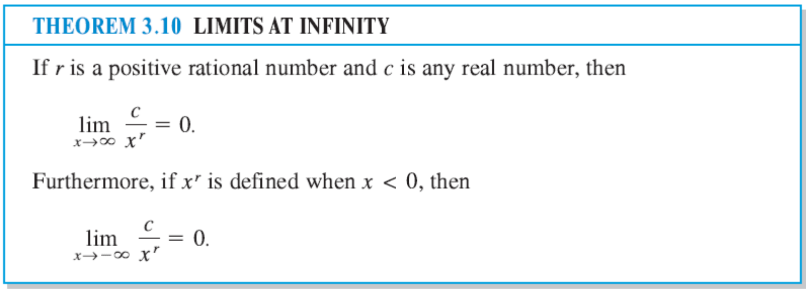
Talk in your group how this definition would relate to graphing functions. How many horizontal asymptotes could you have?

# Properties of Limits to Infinity:

Basically you can break apart a limit to solve it in a quicker way.

$$\lim_{x\to\infty}[f(x)+g(x)] = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$$

$$\lim_{x\to\infty}[f(x)g(x)] = \left[\lim_{x\to\infty} f(x)\right]\left[\lim_{x\to\infty} g(x)\right].$$



**THEOREM 3.10 LIMITS AT INFINITY**

If $r$ is a positive rational number and $c$ is any real number, then

$$\lim_{x\to\infty}\frac{c}{x^r} = 0.$$

Furthermore, if $x^r$ is defined when $x < 0$, then

$$\lim_{x\to-\infty}\frac{c}{x^r} = 0.$$

These are really important make sure you memorize.

example: $\lim_{x\to\infty}\left(5-\frac{2}{x^2}\right) = \lim_{x\to\infty} 5 - \lim_{x\to\infty}\frac{2}{x^2}$

$$5 - 0 = 5$$

example: $\lim_{x\to\infty}\frac{2x-1}{x+1} = \frac{\infty}{\infty}$ indeterminate form

Choice 1 long division

$$\begin{array}{r} 2 \\ x+1\,\overline{)\,2x-1} \\ -\cancel{2x}+2 \\ \hline -3 \end{array} \Rightarrow 2 + \frac{-3}{x+1}$$

$$\lim_{x\to\infty} 2 + \lim_{x\to\infty}\frac{-3}{x+1}$$

$$2 + 0 = \boxed{2}$$

divide by highest degree of x in denominator

Choice 2

$$\lim_{x\to\infty}\frac{\frac{2x-1}{x}}{\frac{x+1}{x}} = \lim_{x\to\infty}\frac{2-\cancel{\frac{1}{x}}^{0}}{1+\cancel{\frac{1}{x}}^{0}} = \frac{2}{1} = \boxed{2}$$

Now in your groups try these 3:

$$\lim_{x\to\infty}\frac{2x+5}{3x^2+1} = \lim_{x\to\infty}\frac{\frac{2x+5}{x^2}}{\frac{3x^2+1}{x^2}} = \frac{\frac{2}{x}+\frac{5}{x^2}}{3+\frac{1}{x^2}} = \frac{0+0}{3+0} = 0$$

$$\lim_{x\to\infty}\frac{2x^2+5}{3x^2+1} = \lim_{x\to\infty}\frac{\frac{2x^2+5}{x^2}}{\frac{3x^2+1}{x^2}} = \frac{2+\frac{5}{x^2}}{3+\frac{1}{x^2}} = \frac{2+0}{3+0} = \frac{2}{3}$$

$$\lim_{x\to\infty}\frac{2x^3+5}{3x^2+1} = \lim_{x\to\infty}\frac{\frac{2x^3+5}{x^2}}{\frac{3x^2+1}{x^2}} = \frac{2x+\frac{5}{x^2}}{3+\frac{1}{x^2}} = \frac{\infty+0}{3+0} = \frac{\infty}{3} = \text{DNE}$$

So Mathematicians saw a pattern with the answers and something else.

Discuss in your groups what that might be.

Lets review the horizontal asymptote rules from Pre-Calculus:

limits are the same

$$y = \frac{ax^m + \cdots}{bx^n + \cdots} \quad if \begin{cases} m < n, & then\ y = 0 \\ m = n, & then\ y = \dfrac{a}{b} \\ m > n, & then\ no\ horizontal\ asy \quad \text{DNE} \end{cases}$$

So Bottom line... the limit to infinity for a rational function is the same as finding the horizontal asymptote.

Unfortunately this only works with rational functions.

What about Irrational functions?

example: $\lim\limits_{x\to\infty}\frac{3x-2}{\sqrt{2x^2+1}}$

note: if $x>0$ then $\sqrt{x^2}=x$

$$\lim_{x\to\infty}\frac{\frac{3x-2}{x}}{\frac{\sqrt{2x^2+1}}{\sqrt{x^2}}}=\frac{\frac{3x}{\cancel{x}}-\frac{2}{x}}{\sqrt{\frac{2x^2}{\cancel{x^2}}+\frac{1}{x^2}}}=\lim_{x\to\infty}\frac{3-\frac{2}{\cancel{x}\,\infty}}{\sqrt{2+\frac{1}{\cancel{x^2}\,\infty}}}=\frac{3-0}{\sqrt{2+0}}=\frac{3}{\sqrt{2}}$$

lim((3x-2)/√(2x^2+1),x,∞) = 2.12132

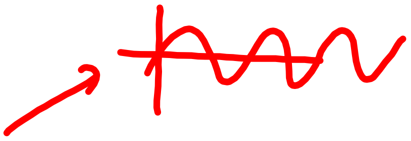
example: $\lim\limits_{x\to-\infty}\frac{3x-2}{\sqrt{2x^2+1}}$

note: if $x<0$ then $\sqrt{x^2}=-x$

$$\lim_{x\to-\infty}\frac{\frac{3x-2}{-x}}{\frac{\sqrt{2x^2+1}}{\sqrt{x^2}}}=\frac{\frac{3x}{-x}+\frac{2}{x}}{\sqrt{\frac{2x^2}{x^2}+\frac{1}{x^2}}}=\lim_{x\to-\infty}\frac{-3+\frac{2}{\cancel{x}\,\infty}}{\sqrt{2+\frac{1}{\cancel{x^2}\,\infty}}}=\frac{-3+0}{\sqrt{2+0}}=\frac{-3}{\sqrt{2}}$$

Limits involving Trig Functions

examples:



1. $\lim\limits_{x\to\infty}\sin x$ = DNE because graph oscillates between 1 and -1

2. $\lim\limits_{x\to\infty}\boxed{\frac{\sin x}{x}}=0$ because we know $\underbrace{-\frac{\overset{0}{\cancel{1}}}{x}<\boxed{\frac{\sin x}{x}}<\frac{\overset{0}{\cancel{1}}}{x}}_{\text{squeeze theorem}}$

$\lim\limits_{x\to\infty}\frac{1}{x}=0$

$\lim\limits_{x\to\infty}-\frac{1}{x}=0$